B.Sc Part I (Maths) Paper III

Concept of a Rings

Definition: Let R be a set and let two binary functions called addition (denoted by +) and multiplication (denoted by ·) be defined over the set R. Then the system $(R, +, \cdot)$ is called a ring R if the following postulates are satisfied

I. Laws of addition

(i) $a+b \in R$, $a, b \in R$

(ii) $(a+b)+c = a+(b+c)$, $a, b, c \in R$ (associative law)

(iii) There exists an element 0 in R called zero of the ring such that $a+0 = 0+a = a$ for every $a \in R$.

(iv) For each element a in R there exists $-a$ in R called the negative of a such that $a+(-a) = -a+a = 0$

(v) $a+b = b+a$, $a, b \in R$ (Commutative law)

II. Laws of multiplication

(i) $a \cdot b \in R$, $a, b \in R$

(ii) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, $a, b, c \in R$ (associative law)

III. Distributive laws.

(i) $a \cdot (b+c) = a \cdot b + a \cdot c$

(ii) $(b+c) \cdot a = b \cdot a + c \cdot a$, $a, b, c \in R$.

Commutative Ring: A ring R is said to be commutative if $a \cdot b = b \cdot a$; $a, b \in R$.

It means that the set R to be a commutative ring must satisfy in addition to the three laws listed above the commutative law for multiplication as well.

1 Ex. Prove that the set of integers is a commutative ring with unit element with respect to usual addition and multiplication.

Solution: — Let I is the set of integers $(0, \pm 1, \pm 2, \pm 3, \ldots)$

We have to show that I is a commutative ring with unit element.

We know from group theory that I is an Abelian group w.r.t. addition.

Secondly, the product of two integers is an integer. i.e. $ab \in I$, $a, b \in I$

Also $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ since $a, b, c \in I$

Thus the laws of multiplication are satisfied.

Thirdly $\left.\begin{array}{l} a \cdot (b+c) = a \cdot b + a \cdot c \\ \text{and } (b+c) \cdot a = b \cdot a + c \cdot a \end{array}\right\} a, b, c \in I$

Thus the distributive laws are satisfied.

Hence the set I is a ring with respect to usual addition and multiplication.

Moreover, I is commutative since $a \cdot b = b \cdot a$, $a, b \in I$

The unit element of I is 1, since $a \cdot 1 = 1 \cdot a = a$ for every $a \in I$.

Thus the set I is commutative ring with unit element 1.

2 Ex-2 Show that a ring R in which $a^2 = a$ for all $a \in R$ is commutative.

Soln: — We have $(a+b) = (a+b)^2 = (a+b)(a+b)$

$= (a+b)a + (a+b)b = a^2 + ba + ab + b^2$

$\Rightarrow a+b = a+b+(ba+ab) \Rightarrow 0 = ba+ab \Rightarrow ab = -ba$ ——(1)

Also $(-a)^2 = -a \Rightarrow (-a)(-a) = -a$

$\Rightarrow a^2 = -a \Rightarrow a = -a \Rightarrow ba = -ba$ ——(2)

Hence from (1) and (2), $ab = ba$ which proves that the ring is commutative.

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